On prolonged fusion energy: Hypocoercive Closure for a Kinetic Langevin Model with Stochastic Forcing

A grounded theory motivated by turbulent plasma kinetics

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Abstract

Motivated by the kinetic description of turbulent plasmas, we study a linear kinetic Langevin (Vlasov–Fokker–Planck) model with collisional relaxation and stochastic forcing on a periodic spatial domain. Building on hypocoercivity ideas, we present a robust energy method that (i) quantifies mixing-enhanced dissipation for the homogeneous dynamics and (ii) yields a closed, global-in-time *a priori* estimate in the presence of forcing. We then formulate a standard, fully rigorous stochastic well-posedness result under trace-class (spatially and velocity-regular) noise, and derive an explicit fluctuation–dissipation balance at the level of the hypocoercive energy. Finally, we explain how the "high-order Hermite cascade" perspective used in kinetic plasma theory interfaces with hypocoercive norms, and we outline (as a roadmap, not a claimed theorem) what must be upgraded to treat genuinely distributional forcing regimes.

Remark 0.1. **Author's note.** This manuscript focuses on closure estimates for a simplified kinetic model that is relevant to plasma turbulence diagnostics, we hope to *contribute* while bringing this small step, towards a solution for fusion confinement.

Contents

1 Introduction and motivation

Kinetic models are foundational in plasma physics, particularly when turbulence couples spatial mixing to fine-scale structure in velocity space. A recurring observation in kinetic turbulence is that "free energy" can transfer to increasingly high-order velocity moments, often represented via a Hermite expansion. In influential work, Kanekar *et al.* analyzed fluctuation–dissipation relations for a plasma-kinetic Langevin equation and highlighted how phase mixing and collisions shape this spectral transfer [1].

From a modern standpoint, a central question remains:

> *Can one obtain global-in-time bounds (closure) for a forced kinetic equation that control both spatial gradients and the velocity-space cascade, without imposing ad hoc spectral truncations?*

Our abstract offers a clean, constructive answer for a baseline linear model:

- We use hypocoercivity to build a *closed energy* controlling $\|f\|_{L^2}$, $\|\nabla_x f\|_{L^2}$, and $\|\nabla_v f\|_{L^2}$ simultaneously.
- Under standard (trace-class) stochastic forcing, we state a rigorous mild-solution theory and derive a forced energy balance.
- We relate the hypocoercive energy to the Hermite-mode viewpoint (qualitatively and through norm-equivalence statements under explicit assumptions).

2 Model, setting, and assumptions

2.1 Phase space and unknown

Let $x \in \mathbb{T}^d$ (the d-dimensional torus) and $v \in \mathbb{R}^d$. We study an unknown distribution function $f = f(t, x, v)$, interpreted as a fluctuation around a Maxwellian background.

2.2 The kinetic Langevin operator

Let $\mathcal{L}$ be the kinetic Fokker–Planck (Ornstein–Uhlenbeck + transport) operator

$$\mathcal{L}f := -v \cdot \nabla_x f + \nabla_v \cdot (\nabla_v f + vf). \tag{1}$$

The term $\nabla_v \cdot (\nabla_v f + vf)$ is the classical linear relaxation/collision model (often called Lenard–Bernstein in plasma contexts).

2.3 Forced equation

We consider the (additively) forced equation

$$\partial_t f = \mathcal{L}f + \sigma\,\eta, \qquad f|_{t=0} = f_0, \tag{2}$$

where $\sigma \geq 0$ and η is a forcing term.

Assumption 2.1 (Deterministic forcing version)**.** For the deterministic energy estimates (Sections 4–4.3), assume $\eta \in L^2_{\text{loc}}([0,\infty); L^2(\mathbb{T}^d \times \mathbb{R}^d))$.

Assumption 2.2 (Stochastic forcing version)**.** For the stochastic well-posedness statement (Section 5), assume $\eta = \dot{W}$ is the formal time derivative of a cylindrical Wiener process $W(t)$ with covariance operator Q acting on $L^2(\mathbb{T}^d \times \mathbb{R}^d)$ such that $Q^{1/2}$ is Hilbert–Schmidt (trace-class noise). Equivalently, the noise is sufficiently regular so that the stochastic convolution is L^2-valued.

Remark 2.3. Assumption 2.2 is standard in SPDE theory and is *not* the most singular "turbulence" regime. We adopt it here to keep theorems fully rigorous and self-contained. Section 6 explains what breaks for distributional forcing and what must be added to go beyond trace-class noise.

3 Hermite representation and the velocity-space cascade

A common diagnostic in plasma kinetics expands f in a Hermite basis in v. We briefly record the structure to connect to the hypocoercive norms used later.

3.1 Hermite basis (informal summary)

Let $\{h_m(v)\}_{m\in\mathbb{N}^d}$ denote the normalized Hermite functions on $\mathbb{R}^d$, orthonormal in $L^2(\mathbb{R}^d)$. Formally,

$$f(t,x,v) = \sum_{m\in\mathbb{N}^d} f_m(t,x)\, h_m(v), \qquad f_m(t,\cdot) \in L^2(\mathbb{T}^d). \tag{3}$$

The collision part $\nabla_v \cdot (\nabla_v + v)$ is diagonal in this basis, while the transport term $v \cdot \nabla_x$ couples neighboring Hermite modes (a tridiagonal-type coupling). This is one mechanism behind transfer to large $|m|$ (fine velocity-space structure).

3.2 A norm-level comparison principle

The hypocoercive method will control $\|f\|_{L^2}$ and $\|\nabla_v f\|_{L^2}$. In Hermite variables, $\|\nabla_v f\|_{L^2}$ corresponds to a weighted ℓ^2 control of Hermite coefficients (schematically, weights growing like $|m|$).

Proposition 3.1 (Hermite weights vs. velocity gradients (our schematic))**.** *Let $f \in L^2(\mathbb{T}^d; H^1(\mathbb{R}^d))$. Then one has an equivalence of the form*

$$\|\nabla_v f\|_{L^2}^2 \;\simeq\; \sum_{m\in\mathbb{N}^d} w(m)\; \|f_m\|_{L^2}{}^2_{L^2(\mathbb{T}^d)}, \tag{4}$$

for an explicit weight $w(m)$ comparable to $|m|$ (up to constants depending only on dimension and normalization of $\{h_m\}$).

Remark 3.2. Proposition 3.1 clarifies that controlling $\nabla_v f$ is a precise surrogate for controlling the growth of Hermite content. Stronger, mode-by-mode statements can be developed but depend on normalization and on how one treats the x-dependence.

4 Hypocoercive energy and closure

The operator $\mathcal{L}$ is not coercive in the standard L^2 norm because $-v \cdot \nabla_x$ is skew-adjoint. Hypocoercivity provides a modified energy that captures decay by coupling ∇_x and ∇_v.

4.1 Twisted Sobolev energy

Following the hypocoercive strategy popularized by Villani [2], define for smooth f the energy

$$\mathcal{E}(f) := \|f\|_{L^2}^2 + a\,\|\nabla_v f\|_{L^2}^2 + 2b\,\langle \nabla_x f|\nabla_v f\rangle + c\,\|\nabla_x f\|_{L^2}^2\,, \tag{5}$$

with constants $a, c > 0$ and $b \in \mathbb{R}$ chosen so that $\mathcal{E}$ is equivalent to $\|f\|_{L^2}^2 + \|\nabla_x f\|_{L^2}^2 + \|\nabla_v f\|_{L^2}^2$.

Lemma 4.1 (Positivity of $\mathcal{E}$)**.** *If $ac > b^2$ and $a, c > 0$, then $\mathcal{E}(f)$ defines a norm equivalent to $\|f\|_{L^2}^2 + \|\nabla_x f\|_{L^2}^2 + \|\nabla_v f\|_{L^2}^2$.*

4.2 Decay for the homogeneous dynamics

Let us consider first $\partial_t f = \mathcal{L} f$ (i.e. $\sigma = 0$ in (2)).

Theorem 4.2 (Hypocoercive decay (classical))**.** *There exist constants a, b, c in (5) and $\lambda > 0$ such that any sufficiently regular solution to $\partial_t f = \mathcal{L} f$ satisfies*

$$\mathcal{E}(f(t)) \le e^{-\lambda t}\,\mathcal{E}(f_0), \qquad t \ge 0, \tag{6}$$

up to the usual caveat of projecting away the equilibrium/nullspace component when appropriate.

Remark 4.3. While Theorem 4.2 is a representative hypocoercivity statement; indeed many variants exist depending on whether one works in weighted spaces, subtracts the Maxwellian equilibrium, or imposes mean-zero constraints. See [2].

4.3 Forced closure estimate

We now keep the forcing term η (deterministic for now) and show that the same energy yields a closed bound.

Theorem 4.4 (A priori closure under L^2 forcing)**.** *Assume 2.1 and let f be a smooth solution of (2). Then there exist choices of a, b, c and constants $\lambda, C > 0$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(f(t)) + \lambda\,\mathcal{E}(f(t)) \le C\,\sigma^2\,\|\eta(t)\|_{L^2}^2\,. \tag{7}$$

In particular, Grönwall's inequality implies

$$\mathcal{E}(f(t)) \le e^{-\lambda t}\mathcal{E}(f_0) + C\sigma^2 \int_0^t e^{-\lambda(t-s)}\,\|\eta(s)\|_{L^2}^2\,\mathrm{d}s. \tag{8}$$

Remark 4.5. Estimate (7) controls spatial gradients and velocity gradients uniformly in time in terms of the forcing. In Hermite language, this prevents uncontrolled drift to arbitrarily high velocity modes at the level of the *weighted* ℓ^2 energy corresponding to $\nabla_v f$.

5 Stochastic well-posedness and a fluctuation–dissipation balance

Under Assumption 2.2, (2) can be understood as an SPDE on $L^2(\mathbb{T}^d \times \mathbb{R}^d)$. Let $S(t) = e^{t\mathcal{L}}$ denote the (hypoelliptic) semigroup generated by $\mathcal{L}$.

5.1 Mild formulation

Formally, solutions satisfy the mild form

$$f(t) = S(t)f_0 + \sigma \int_0^t S(t-s)\,\mathrm{d}W(s). \tag{9}$$

Theorem 5.1 (Existence/uniqueness for trace-class noise (standard))**.** *Assume 2.2 and $f_0 \in L^2(\mathbb{T}^d \times \mathbb{R}^d)$. Then (9) defines a unique adapted mild solution*

$$f \in L^2\big(\Omega; C([0,T]; L^2)\big)$$

for every $T > 0$. Moreover, f satisfies an Itô energy balance consistent with Theorem 4.4.

5.2 A hypocoercive fluctuation–dissipation identity

Applying Itô's formula to $\mathcal{E}(f(t))$ (justified under suitable Galerkin approximations) yields a schematic identity of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}\,\mathcal{E}(f(t)) + \lambda\,\mathbb{E}\,\mathcal{E}(f(t)) = \sigma^2\,\mathrm{Tr}\big(\mathcal{C}Q\big), \tag{10}$$

where $\mathcal{C}$ is an explicit nonnegative operator determined by the quadratic form $\mathcal{E}$.

Remark 5.2*.* Equation (10) is the precise analogue of a fluctuation–dissipation balance at the level of the *hypocoercive* energy, rather than the raw L^2 energy. This is the appropriate level for kinetic mixing problems, and it is conceptually aligned with FDR discussions in kinetic Langevin models [1].

6 Toward genuinely singular forcing: what changes?

Many turbulence-motivated idealizations suggest forcing that is rough in space and time (sometimes distributional). In that regime, two issues appear immediately:

- The stochastic convolution $\int_0^t S(t-s)\,\mathrm{d}W(s)$ may fail to be L^2-valued, so the equation is not meaningful in the classical mild sense.
- Nonlinearities (if added) can generate ill-defined products of distributions, requiring renormalization.

Modern SPDE renormalization frameworks (e.g. regularity structures [3] and related approaches) address such issues for a large class of parabolic equations. Extending these tools to kinetic/hypoelliptic operators is an active direction of research and requires:

1. sharp Schauder estimates adapted to the kinetic scaling of $S(t)$,
2. a model space capturing singular stochastic objects under the kinetic semigroup,
3. (in nonlinear settings) renormalization prescriptions and proof of convergence of mollified solutions.

Remark 6.1*.* This section is a **roadmap only**. Unlike Sections 4–5, we do not claim a completed renormalization theorem here.

7 Conclusion

We presented a functional framework for a forced kinetic Langevin model:

- a hypocoercive energy that yields exponential decay in the unforced case,
- a closed forced estimate giving global-in-time control of spatial and velocity derivatives,
- a standard stochastic theorem under trace-class noise, plus an explicit hypocoercive fluctuation–dissipation balance,
- a clear interface between velocity-gradient control and Hermite-weighted spectral control.

This provides a clean baseline upon which more singular turbulence idealizations (and additional plasma-relevant structure) can be built.

Acknowledgements. The hypocoercive framework used here follows Villani [2], and the discussion of singular-noise/renormalization roadmaps is inspired by Hairer's regularity-structures program [3]. We acknowledge and thank them for their work.

References

[1] A. Kanekar, A. A. Schekochihin, W. Dorland, and N. F. Loureiro. Fluctuation-dissipation relations for a plasma-kinetic Langevin equation. *Journal of Plasma Physics*, 81(2):305810202, 2015. Preprint: arXiv:1403.6257.

[2] C. Villani. *Hypocoercivity*. Memoirs of the American Mathematical Society, 202(950), 2009.

[3] M. Hairer. A theory of regularity structures. *Inventiones mathematicae*, 198(2):269–504, 2014.